Key 2015

Advanced Derivative Review

1. f(x), g(x) and h(x) are all differentiable functions with selected values given:

x	f(x)	g(x)	f'(x)	g'(x)
1	2	5	2	3
2	4	1	4	5
4	5	2	6	7

Find the numerical values of h'(2) for each of the following:

a. $h(x)=f(x)g(x)$ $h'(x)=f(x)g'(x)+g(x)f'(x)=4\cdot5+1\cdot4=24$

b. $h(x)=f(g(x))$ $h'(x)=f'(g(x))g'(x)=f'(g(2))g'(2)=f'(1)g'(2)=2\cdot5=10$

c. $h(x)=g(f(x))$ $h'(x)=g'(f(x))\cdot f'(x)=g'(f(2))\cdot f'(2)=g'(4)\cdot4=7\cdot4=28$

d. $h(x)=\dfrac{f(x)}{g(x)}$ $h'(x)=\dfrac{gf'-fg'}{g^2}=\dfrac{1\cdot4-4\cdot5}{1^2}=-16$

e. $h(x)=\sqrt{f(2x)}$ $h'(x)=\frac{1}{2}(f(2x))^{-1/2}\cdot f'(2x)\cdot2=\frac{1}{2}(f(4))^{-1/2}\cdot f'(4)\cdot2=\frac{1}{2\sqrt{5}}\cdot6\cdot2$

$u=f(2x)$ $u'=f'(2x)\cdot2$ $=\dfrac{6}{\sqrt{5}}$

f. $h(x)=[g(x)]^2$ $h'(x)=2g(x)g'(x)=2g(2)g'(2)=2\cdot1\cdot5=10$

g. $h(x)=\ln(f(x))$ $h'(x)=\dfrac{1}{f(x)}\cdot f'(x)=\dfrac{1}{f(2)}\cdot f'(2)=\dfrac{1}{4}\cdot4=1$:)

2. Consider the curve given by $2y^3+6x^2y-12x^2+6y=1$.

a. Find dy/dx

$6x^2|y$
$12x$ dy/dx

$6y^2\frac{dy}{dx}+6x^2\frac{dy}{dx}+12xy-24x+6\frac{dy}{dx}=0$

$6y^2\frac{dy}{dx}+6x^2\frac{dy}{dx}+6\frac{dy}{dx}=24x-12xy$

$\boxed{\dfrac{dy}{dx}=\dfrac{24x-12xy}{6y^2+6x^2+6} \text{ or } \dfrac{dy}{dx}=\dfrac{4x-2xy}{y^2+x^2+1}}$

b. Write the equation of each horizontal tangent to the curve.

$0=24x-12xy$
$0=12x(2-y)$
$x=0$ $y=2$

$\boxed{y=2}$

if $x=0$ $2y^3+6y=1$
$\boxed{y\approx0.165}$ by calculator

3. Given $x^2+y^3=1$, Find $\dfrac{d^2y}{dx^2}$ at $(3,-2)$

$\dfrac{1}{12}$

$2x+3y^2\frac{dy}{dx}=0$

$3y^2\frac{dy}{dx}=-2x$

$\frac{dy}{dx}=\dfrac{-2x}{3y^2}$

$\dfrac{d^2y}{dx^2}=\dfrac{3y^2(-2)-(-2x)6y\frac{dy}{dx}}{9y^4}$

$=\dfrac{-6y^2+12xy\left(\frac{-2x}{3y^2}\right)}{9y^4}$

$=\dfrac{-6y^2-\frac{8x^2}{y}}{9y^4}\Bigg|_{(3,-2)}=\dfrac{-6(4)-\frac{8(9)}{-2}}{9(16)}$

$\dfrac{-24+36}{144}=\boxed{\dfrac{1}{12}}$

4. Find the derivative of the following functions:

$u = 5x \quad u' = 5$

a. $y = \sec(5x) - \csc(5x)$ $\quad y' = 5\sec(5x)\tan(5x) - (-5)\csc(5x)\cot(5x)$

$= \boxed{5\sec(5x)\tan(5x) + 5\csc(5x)\cot(5x)}$

simplified on paper b. $f(x) = \dfrac{\sqrt{5+x^2}}{x^4+1}$ $\quad f'(x) = \dfrac{(x^4+1)(1/2)(5+x^2)^{-1/2}(2x) - (5+x^2)^{1/2}(4x^3)}{(x^4+1)^2}$

$u = \sin(5x)$ c. $f(x) = \sin^2(5x)$ $\quad f'(x) = 2\sin(5x) \cdot \cos 5x \cdot 5 = \boxed{10\sin(5x)\cos(5x)}$

$u' = 5\cos(5x)$ $\quad f(x) = (\sin(5x))^2$

d. $h(x) = \dfrac{5}{(5x^2+3)}$ $\quad h'(x) = \dfrac{(5x^2+3)(0) - 5(10x)}{(5x^2+3)^2} = \boxed{\dfrac{-50x}{(5x^2+3)^2}}$

or $5(5x^2+3)^{-1}$

e. $\sin\sqrt[3]{x}$ $\quad \frac{d}{dx}\sin x^{1/3} = \boxed{\cos x^{1/3} \cdot (\tfrac{1}{3}x^{-2/3})}$ or $\dfrac{\cos x^{1/3}}{3x^{2/3}}$

$\sin x^{1/3}$

f. 7^{3x^2+4x} $= \boxed{\ln 7\,(7^{3x^2+4x})(6x+4)}$ Yuk!

$u = e^{2x}$

$u' = 2e^{2x}$

g. $y = \tan(e^{2x})$ $\quad \boxed{y' = \sec^2(e^{2x}) \cdot 2e^{2x}}$

h. $y = \sin^{-1}(2x)$ $\quad y' = \dfrac{1}{\sqrt{1-4x^2}} \cdot 2 = \boxed{\dfrac{2}{\sqrt{1-4x^2}}}$

$u = 2x \quad u' = 2$

$u = 5x^4$

$u' = 20x^3$

i. $y = \ln(5x^4)$ $\quad y' = \dfrac{1}{5x^4} \cdot 20x^3 = \boxed{\dfrac{4}{x}}$ $\quad$ or $\ln 5 + 4\ln x \quad y' = \boxed{\dfrac{4}{x}}$

j. $y = e^{-4x}$ $\quad \boxed{y' = -4e^{-4x}}$

$u = -4x$

$u' = -4$

Yuk

$u = \tan(\ln(1+x))$ k. $w(x) = \tan^2(\ln(1+x))$ $\quad \boxed{w'(x) = 2\tan(\ln(1+x)) \cdot \sec^2(\ln(1+x)) \cdot \frac{1}{1+x}}$

$(\tan(\ln(1+x)))^2$

$u' = \sec^2(\ln(1+x)) \cdot \frac{1}{1+x}$

$u = \ln(3x)$

$u' = \frac{1}{3x} \cdot 3 = \frac{1}{x}$

l. $y = \cos(\ln(3x))$ $\quad y' = -\sin(\ln(3x)) \cdot \frac{1}{x}$ or $\boxed{\dfrac{-\sin(\ln(3x))}{x}}$

m. $y = t^2\cos^{-1}(2x-1)$ $\quad y' = \dfrac{-2x^2}{\sqrt{1-(2x-1)^2}} + 2x\cos^{-1}(2x-1)$

see paper

n. $\sin(xy) = x$ $\quad dy/dx = \dfrac{1 - y\cos(xy)}{x\cos(xy)}$

see paper

o. $y = x(y+1)$

$\frac{dy}{dx} = x\left(\frac{dy}{dx}\right) + (y+1)$

$\frac{dy}{dx} - x\frac{dy}{dx} = y+1$

$\frac{dy}{dx} = \boxed{\dfrac{y+1}{1-x}}$

Find y''

$y'' = \dfrac{(1-x)\left(\frac{dy}{dx}\right) - (y+1)(-1)}{(1-x)^2}$

$y'' = \dfrac{(1-x)\left(\frac{y+1}{1-x}\right) + y+1}{(1-x)^2}$

$\boxed{y'' = \dfrac{2y+2}{(1-x)^2}}$

$u = -x^2$
$u' = -2x$

$u = \sec x$
$u' = \sec x \tan x$

$f'(x) = \sec^2 x + e^{-x^2} \cdot (-2x) = (\sec x)^2 - 2x e^{-x^2}$

5. If $f(x)$ $y = \tan x + e^{-x^2}$ then f''(0)= ?

$f''(x) = 2\sec x(\sec x \tan x) - (2x e^{-x^2} \cdot -2x + e^{-x^2} \cdot 2)$

$2x \mid e^{-x^2}$
$2 \quad e^{-x^2} \cdot -2x$

$f''(0) = 2\sec 0 (\sec 0 \tan 0) - (0 + e^0 \cdot 2)$

$f''(0) = -e^0 \cdot 2 = -2$

6. If $f(x) = \dfrac{e^{-x^2}}{x^2+1}$, then f'(1)= ?

$f'(x) = \dfrac{(x^2+1)(e^{-x^2})(-2x) - e^{-x^2}(2x)}{(x^2+1)^2}$ $\quad f'(1) = \dfrac{(1+1)(e^{-1})(-2) - e^{-1}(2)}{(1+1)^2} = \dfrac{-4e^{-1} - 2e^{-1}}{4} = \dfrac{-6e^{-1}}{4} = \boxed{\dfrac{-3}{2e}}$

7. Given: $y = \sin^2 x$. Find the equation for the tangent and normal lines to the graph where $x = \dfrac{\pi}{6}$.

$y = (\sin x)^2$ $\quad y' = 2\sin x \cdot \cos x$

$u = \sin x$
$u' = \cos x$

$y'(\pi/6) = 2\sin \pi/6 \cos \pi/6 = 2(1/2)\left(\frac{\sqrt{3}}{2}\right) = \boxed{\frac{\sqrt{3}}{2}}$

$(\pi/6, 1/4)$

$y = (\sin^2 \frac{\pi}{6})^2$
$y = (1/2) = 1/4$

tang: $y - 1/4 = \sqrt{3}/2 (x - \pi/6)$
$\perp$ normal: $y - 1/4 = \frac{-2}{\sqrt{3}}(x - \pi/6)$

8. Discuss the continuity and differentiability of

$f(x) = \begin{cases} 8x-3, & x \le 1 \\ 4x^2+1, & x > 1 \end{cases}$ at x=1

at $x=1$ $\begin{cases} 8-3=5 \\ 4+1=5 \end{cases}$ $\lim_{x \to 1} f(x)$ exists

$f(1) = 5$

$\lim_{x\to 1} f(x) = f(1)$ $\therefore$ continuous at $x=1$

$f'(x) = \begin{cases} 8 & x \le 1 \\ 8x & x > 1 \end{cases}$

at $x = 1$

$\lim_{x \to 1^+} f'(x) = \lim_{x \to 1^-} f'(x)$

$\therefore$ Differentiable at $x=1$

9. A particle moves along the x axis with velocity at time $t \ge 0$ given by $v(t) = -1 + e^{1-t}$ with t given in seconds and v(t) in ft/sec.

a. Find the acceleration at t=3. $v'(t) = a(t) = e^{(1-t)} \cdot (-1)$

$a(3) = e^{1-3} \cdot -1 = -e^{-2}$ or $\boxed{\frac{-1}{e^2}}$ ft/sec^2

b. Is the speed of the particle increasing or decreasing at t=3? Justify.

$a < 0$ from part a

$v(3) = -1 + e^{1-3}$
$v(3) = -1 + \frac{1}{e^2}$ $\quad v(3) < 0$

speed is increasing since $v<0$ and $a<0$ at $t=3$.

c. Find all values of t where the particle changes direction. Justify. use v.

$0 = -1 + e^{1-t}$
$1 = e^{1-t}$

$\ln 1 = \ln e^{1-t}$
$\cancel{\ln 1} = \cancel{\ln e}(1-t)$
$0 \qquad 1$

$0 = 1 - t$
$t = 1$

$+ \mid -$
1

at $t=1$ $v(t)$ changes sign so particle changes direction

10. Find y'' if $y = e^{2x} \ln(x)$.

$2 \cdot e^{2x} \mid \frac{1}{x}$

$y' = e^{2x} \cdot \frac{1}{x} + \ln x \cdot 2e^{2x} = e^{2x}\left(\frac{1}{x} + 2\ln x\right)$

$\boxed{y'' = e^{2x}\left(\frac{-1}{x^2} + \frac{4}{x} + 4\ln x\right)}$

see paper

11. Given that $g(2) = 5$, $f'(2) = 5$ and $f'(5) = \dfrac{1}{2}$. If f and g are inverses to each other, then

g: 2,5
f: 5,2

$g'(2) = \dfrac{1}{1/2} = 2$

$g'(2) = \dfrac{1}{f'(5)}$

12. Let $f(x) = x^3 + x$. If h is the inverse function of f, and f(1)=2, then $h'(2) =$

(1,2)
h(2,1)

$f'(x) = 3x^2 + 1$
$f'(1) = 3 + 1 = 4$

$h'(2) = \dfrac{1}{f'(1)}$

$h'(2) = \boxed{\dfrac{1}{4}}$

13. Find $\frac{d}{dt}$ of the following equations.

a. $V = \frac{4}{3}\pi r^3$

$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$

b. $A = \frac{1}{2}bh$ db/dt dh/dt

$\frac{dA}{dt} = \frac{1}{2}b\frac{dh}{dt} + \frac{1}{2}h\frac{db}{dt}$

c. $S = 2\pi rh$ $2\pi\frac{dr}{dt}$, dh/dt

$\frac{dS}{dt} = 2\pi r\frac{dh}{dt} + h(2\pi)\frac{dr}{dt}$

14. Two zombies travel in perpendicular directions. one towards one away. Zombie Y From an intersection travels due south at a rate of 3 miles per hour; Zombie X travels due west at a rate of 4 miles per hour. At what rate is the distance between the zombies changing at $x=8$, $y=6$

$\frac{dy}{dt} = -3\text{mph}$ $y=6$

$\frac{dx}{dt} = 4\text{mph}$ $x=8$

$d=10$

$d^2 = x^2 + y^2$

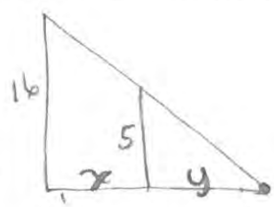
$2d\frac{dd}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$

$20\frac{dd}{dt} = 16\cdot 4 + (12)(-3)$

$20\frac{dd}{dt} = 28$

$\frac{dd}{dt} = \frac{7}{5}\text{mph}$

15. A zombie 5 feet tall walks[shambles] at a rate of 3 feet per minute away from a light that is 16 feet above the ground. When the zombie is 10 feet from the base of the light, what is the rate at which ~~the tip of~~ the shadow increases is moving? Find $\frac{dy}{dt}$ similar Δ's



$x=10$ $\frac{dx}{dt} = 3\text{ft/min}$

$\frac{16}{x+y} = \frac{5}{y}$

$16y = 5x + 5y$

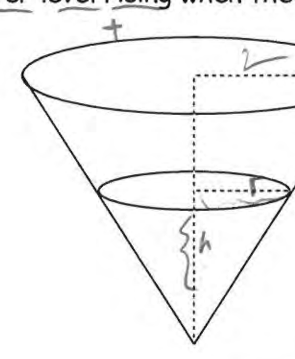
$11y = 5x$

$y = \frac{5x}{11}$

$\frac{dy}{dt} = \frac{5}{11}\frac{dx}{dt}$

$\frac{dy}{dt}$? $= \boxed{\frac{15}{11}\frac{\text{ft}}{\text{min}}}$

16. Water runs into a conical tank at a the rate of 3 ft^3/min. The tank stands point down and has a height of 5 ft and a base radius of 2 ft. How fast is the water level rising when the water is 3 ft. deep? dh/dt?



when $h=3$

$\frac{dV}{dt} = 3\text{ft}^3/\text{min}$

Don't know dr/dt so must eliminate r

$\frac{2}{5} = \frac{r}{h}$

$2h = 5r$

$\frac{2h}{5} = r$

$V = \frac{1}{3}\pi r^2 h$

$V = \frac{1}{3}\pi\left(\frac{2h}{5}\right)^2 h$

$V = \frac{1}{3}\pi\frac{4}{25}h^3$

$V = \frac{4}{75}\pi h^3$

$\frac{dV}{dt} = \frac{4}{75}\pi\cdot 3h^2\frac{dh}{dt}$

$3 = \frac{4}{25}\pi(9)\frac{dh}{dt}$

$\frac{75}{36\pi} = \frac{dh}{dt}$

$h=6$

$\frac{dh}{dt} = \boxed{.663\,\text{ft/min}}$

or $\frac{25}{36\pi}$

You made it to the end!

Key

CALCULUS BC
WORKSHEET ON PARAMETRICS AND CALCULUS

Work these on **notebook paper**. Do not use your calculator.

On problems 1 – 5, find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

1. $x = t^2,\ y = t^2 + 6t + 5$

4. $x = \ln t,\ y = t^2 + t$

~~2. $x = t^2 + 1,\ y = 2t^3 - t^2$~~

~~5. $x = 3\sin t + 2,\ y = 4\cos t - 1$~~

~~3. $x = \sqrt{t},\ y = 3t^2 + 2t$~~

1) $\frac{dx}{dt} = 2t \quad \frac{dy}{dt} = 2t+6$

$\frac{dy}{dx} = \frac{2t+6}{2t} = \frac{t+3}{t}$

$\frac{d^2y}{dx^2} = \frac{\frac{t(1)-(t+3)}{t^2}}{2t} = \boxed{\frac{-3}{2t^3}}$

(4) $\frac{dx}{dt} = \frac{1}{t} \quad \frac{dy}{dt} = 2t+1$

$\frac{dy}{dx} = \frac{2t+1}{\frac{1}{t}} = \boxed{2t^2+t}$ $\quad \frac{d^2y}{dx^2} = \boxed{\frac{4t+1}{2t}}$

6. A curve C is defined by the parametric equations $x = t^2 + t - 1,\ y = t^3 - t^2$.

(a) Find $\frac{dy}{dx}$ in terms of t.

$\frac{dx}{dt} = 2t+1 \quad \frac{dy}{dt} = 3t^2 - 2t \quad \frac{dy}{dx} = \frac{3t^2-2t}{2t+1}$

(b) Find an equation of the tangent line to C at the point where $t = 2$.

$x = 4+2-1 = 5 \quad y = 8-4 = 4 \quad (5,4)$

$\left.\frac{dy}{dx}\right|_{t=2} = \frac{12-4}{4+1} = \frac{8}{5}$

$\boxed{y-4 = \frac{8}{5}(x-5)}$

7. A curve C is defined by the parametric equations $x = 2\cos t,\ y = 3\sin t$.

(a) Find $\frac{dy}{dx}$ in terms of t.

$\frac{dx}{dt} = -2\sin t \quad \frac{dy}{dt} = 3\cos t \quad \boxed{\frac{dy}{dx} = \frac{3\cos t}{-2\sin t}}$

$\frac{dy}{dx} = \frac{3\frac{\sqrt{2}}{2}}{-2\frac{\sqrt{2}}{2}} = -3/2$

(b) Find an equation of the tangent line to C at the point where $t = \frac{\pi}{4}$.

$x = 2\cos \pi/4 = 2\sqrt{2}/2 \quad y = 3\cos \pi/4 = 3\sqrt{2}/2$

$\boxed{y - 3\sqrt{2}/2 = -3/2(x - \sqrt{2})}$

On problems 8 – 10, find:

(a) $\frac{dy}{dx}$ in terms of t.

(b) all points of horizontal and vertical tangency

8. $x = t + 5,\ y = t^2 - 4t$

9. $x = t^2 - t + 1,\ y = t^3 - 3t$

10. $x = 3 + 2\cos t,\ y = -1 + 4\sin t$

(8) $\frac{dx}{dt} = 1 \quad \frac{dy}{dt} = 2t-4 \quad \frac{dy}{dx} = \frac{2t-4}{1}$

HT: $2t-4=0$

$2t = 4$

$t = 2$

$x = 7 \quad y = 0$

$(7,0)$

no VT.

On problems 11 - 12, a curve C is defined by the parametric equations given. For each problem, write an integral expression that represents the length of the arc of the curve over the given interval.

11. $x = t^2,\ y = t^3,\ 0 \le t \le 2$

12. $x = e^{2t} + 1,\ y = 3t - 1,\ -2 \le t \le 2$

$\frac{dx}{dt} = 2t-1 \quad \frac{dy}{dx} = \frac{3t^2-3}{2t-1}$

$\frac{dy}{dt} = 3t^2 - 3$

VT: $2t-1=0$

$t = 1/2$

$x = \frac{1}{4} - \frac{1}{2} + 1 = 3/4$

$y = 1/8 - 3/8 = -1/4$

$(3/4, -1/4)$

HT: $3t^2 - 3 = 0$

$t = \pm 1$

$t=1:\ x=1 \quad y=-2 \quad (1,-2)$

$t=-1:\ x=3 \quad y=2 \quad (3,2)$

4.b. $f'(x) = \dfrac{(x^4+1)(\frac{1}{2})(5+x^2)^{-1/2}(2x) - (5+x^2)^{1/2}(4x^3)}{(x^4+1)^2}$

$f'(x) = \dfrac{(5+x^2)^{-1/2}\left[(x^4+1)(x) - (5+x^2)(4x^3)\right]}{(x^4+1)^2}$

$x^5 + x - 20x^3 - 4x^5$

$$\boxed{f'(x) = \frac{-3x^5 - 20x^3 + x}{(x^4+1)^2(5+x^2)^{1/2}}}$$

4m. $y = x^2 \cos^{-1}(2x-1)$

$$\begin{array}{c|c} x^2 & \cos^{-1}(2x-1) \\ 2x & -\frac{1}{\sqrt{1-(2x-1)^2}} \cdot 2 \end{array} \quad \begin{array}{l} u = 2x-1 \\ u' = 2 \end{array}$$

$y' = x^2 \cdot \dfrac{-2}{\sqrt{1-(2x-1)^2}} + \cos^{-1}(2x-1) \cdot 2x$

4n. $\sin(xy) = x$

$$\begin{array}{c|c} x & y \\ 1 & \frac{dy}{dx} \end{array}$$

$\cos(xy)(x\frac{dy}{dx} + y) = 1$

$\cos(xy)\, x\frac{dy}{dx} + y\cos(xy) = 1$

$x\cos(xy)\frac{dy}{dx} = 1 - y\cos(xy)$

$\dfrac{dy}{dx} = \dfrac{1 - y\cos(xy)}{x\cos(xy)}$

10. $y' = e^{2x} \cdot \frac{1}{x} + \ln(x) \cdot 2e^{2x}$

factor $y' = e^{2x}\left(\frac{1}{x} + 2\ln(x)\right)$

$$\begin{array}{c|c} 2e^{2x} & -1x^{-2} + 2/x \end{array}$$

$y'' = e^{2x}\left(\frac{-1}{x^2} + \frac{2}{x}\right) + \left(\frac{1}{x} + 2\ln x\right)2e^{2x}$

$y'' = e^{2x}\left(\frac{-1}{x^2} + \frac{2}{x} + \frac{2}{x} + 4\ln x\right)$

$y'' = e^{2x}\left(-\frac{1}{x^2} + \frac{4}{x} + 4\ln x\right)$